TOPOLOGICAL MINIMAL GENUS AND L^2-SIGNATURES

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Abstract. We obtain new lower bounds of the minimal genus of a locally flat surface representing a 2-dimensional homology class in a topological 4-manifold with boundary, using the von Neumann-Cheeger-Gromov ρ-invariant. As an application our results are employed to investigate the slice genus of knots. We illustrate examples with arbitrarily large slice genus for which our lower bound is optimal but all previously known invariants vanish.

1. Introduction and main results

This paper concerns the problem of the minimal genus of a locally flat embedded surface representing a given 2-dimensional homology class in a topological 4-manifold. Precisely, a locally flat closed surface Σ in a topological 4-manifold W is said to represent $\sigma \in H_2(W)$ if the fundamental class of Σ is sent to σ under the map induced by the inclusion. In this paper manifolds are always oriented and surfaces are assumed to be connected.

For topological 4-manifolds which are closed (or with boundaries consisting of homology spheres), there are remarkable known results which provide lower bounds of the minimal genus, including Kervaire-Milnor [KM61], Hsiang-Szczarba [HS71], Rokhlin [Rok71], and Lee-Wilczyński [LW97, LW00]. Basically these lower bounds are extracted by considering Rokhlin's theorem and algebraic topology of finite cyclic branched coverings. Also, interesting results on the smooth analogue of this problem have been obtained using gauge theory for a certain class of smooth 4-manifolds. Related works include results on the Thom conjecture and adjunction inequality due to Kronheimer-Mrowka [KM93, KM95b, KM95a], Morgan-Szabó-Taubes [MST96], Kronheimer [Kro99], and Ozsváth-Szabó [OS00b, OS00a]. One may obtain a lower bound in a 4-manifold with boundary when it embeds into another 4-manifold for which the above lower bound results can be applied directly. For example, the adjunction inequality is proved in Stein 4-manifolds via an embedding theorem due to Lisca-Matić [LM97, LM98, AM97].

In this paper we focus on the minimal genus problem in a topological 4-manifold which has boundary with nontrivial homology. Our results give new lower bounds of the minimal genus, for homology classes from the boundary, in terms of the von Neumann-Cheeger-Gromov ρ-invariants of the boundary. As an application we give lower bounds of the slice genus of a knot. Examples illustrate that the ρ-invariants detect arbitrarily large minimal genus that all previously known results do not.

Minimal second Betti number of a 4-dimensional bordism. We obtain lower bounds of the minimal genus through the following problem on 4-dimensional bordisms: what is the minimal second Betti number of a topological null-bordism of a given closed 3-manifold endowed with a group homomorphism of the fundamental group? Our principal result on this is as follows. Let Γ be a poly-torsion-free-abelian (PTFA) group, i.e., Γ admits a finite length normal series $\{G_i\}$ with

2000 *Mathematics Subject Classification.* 57N13, 57N35, 57R95, 57M25.

Key words and phrases. 4-manifold, Minimal Genus, Minimal Betti Number, Slice Genus, L^2-signature.

G_i/G_{i+1} torsion-free abelian. It is known that there is a (skew-)field $\mathcal{K}$ of right quotients of $\mathbb{Z}\Gamma$. Let $\mathcal{R}$ be a subring of $\mathcal{K}$ which is a PID containing $\mathbb{Z}\Gamma$. Then Γ acts on the abelian group $\mathcal{K}/\mathcal{R}$ via right multiplication so that the semi-direct product $(\mathcal{K}/\mathcal{R}) \rtimes \Gamma$ is defined.

Theorem 1.1. *Suppose M is a closed 3-manifold endowed with a homomorphism $\phi\colon \pi_1(M) \to \Gamma$, and W is a topological 4-manifold with boundary M such that ϕ factors through $\pi_1(W)$. Then the followings hold:*

(1) *The second Betti number $\beta_2(W)$ satisfies*

$$|\rho(M, \phi)| \le 2\beta_2(W)$$

where $\rho(M, \phi) \in \mathbb{R}$ denotes the von Neumann-Cheeger-Gromov ρ-invariant of M associated to ϕ.

(2) *In addition, if the twisted homology $H_1(M; \mathcal{R})$ is $\mathcal{R}$-torsion and not generated by any $\beta_2(W)$ elements, then there is a nontrivial submodule P in* $\operatorname{Hom}(H_1(M; \mathcal{R}), \mathcal{K}/\mathcal{R})$ *such that every homomorphism in P gives rise to a lift $\phi_1 \colon \pi_1(M) \to (\mathcal{K}/\mathcal{R}) \rtimes \Gamma$ of ϕ which factors through $\pi_1(W)$.*

More detailed versions of Theorem 1.1 (1) and (2) are stated and proved in Section 2 and 3, respectively. To prove (1), we regard the ρ-invariant of M as an L^2-signature defect of W, and estimate the L^2-signature of W in terms of the L^2 and ordinary Betti number. While (1) gives a lower bound of $\beta_2(W)$ without using (2), further information may be obtained when (2) is combined with (1); note that (2) gives a sufficient condition which implies that a certain "bigger" coefficient system of M, namely ϕ_1, extends to W. In case that ϕ_1 extends, (1) can be applied again to ϕ_1 to obtain further lower bounds of $\beta_2(W)$ (and possibly this process may be iterated).

This type of coefficient extension problem plays a crucial role in earlier landmark works in knot theory, including Casson and Gordon [CG86, CG78], Gilmer [Gil82], and in particular Cochran, Orr, and Teichner [COT03, COT04], from which Theorem 1.1 has been directly motivated. In [COT03, COT04] the extension problem is investigated when $H_1(\partial W; \mathbb{Q}) \cong H_1(W; \mathbb{Q}) \cong \mathbb{Q}$ and W satisfies some geometric condition related to the existence of a Whitney tower (such W is called an (h)-solution in [COT03]). In order to deal with the extension problem without assuming these conditions, as in Theorem 1.1 (2), we investigate the relationship of the Blanchfield linking form of M and the intersection form of W over $\mathcal{R}$-coefficients, and import ideas from Gilmer's work [Gil82] on Casson-Gordon invariants.

The following result relates the minimal second Betti number of bordisms with a particular type of the minimal genus problem in a 4-manifold with boundary. Suppose W is a topological 4-manifold with boundary M, $H_1(W) = 0$, and σ is a 2-dimensional homology class contained in the image of $H_2(M) \to H_2(W)$. In Section 4 we will describe a homomorphism $\phi_\sigma \colon \pi_1(M) \to \mathbb{Z}$ determined by σ.

Proposition 1.2. *If ϕ_σ is nontrivial and there is a locally flat embedded surface of genus g in W representing σ, then there is a topological 4-manifold V bounded by M such that $\phi_\sigma \colon \pi_1(M) \to \mathbb{Z}$ factors through $\pi_1(V)$ and $\beta_2(V) = \beta_2(W) + 2g - 1$.*

Consequently, lower bounds of $\beta_2(V)$ obtained by (possibly repeatedly) applying Theorem 1.1 give rise to lower bounds of the genus g.

Slice genus of a knot. As an application, we employ our results on the minimal genus problem to investigate the slice genus of a knot K in S^3. The *topological slice genus* $g^t_*(K)$ of K is defined to be the minimal genus of a locally flat surface F properly embedded in D^4 in such a way that $\partial F = K$, viewing S^3 as the boundary

of D^4. The *smooth slice genus* $g_*^s(K)$ is defined similarly, requiring F to be a smooth submanifold of D^4. Obviously $g_*^t(K) \le g_*^s(K)$.

There are various known lower bounds of the slice genus. Clearly any obstruction to being a slice knot can be viewed as a lower bound of the form (slice genus) ≥ 1. It is well known that some invariants derived from a Seifert matrix, including the signature of a knot, can be used to detect higher topological slice genus. Gilmer showed that Casson-Gordon invariants of a knot K give further lower bounds of $g_*^t(K)$ [Gil82]. For the smooth slice genus, further results based on gauge theory are known. For a special class of knots which includes the torus knots, an optimal lower bound is obtained as an application of the Thom conjecture due to Mrowka-Kronheimer [KM93]. For an arbitrarily given knot K, the Thurston-Bennequin invariant (together with the rotation invariant) of a Legendrian representation of K is known to give a lower bound of $g_*^s(K)$, due to Rudolph [Rud95, Rud97], Kronheimer-Mrowka, Akbulut-Matveyev [AM97], and Lisca-Matić [LM97, LM98]. More recently, Ozsváth-Szabó's τ-invariant [OS03] and Rasmussen's s-invariant [Ras04] defined from knot homology theories of Ozsváth-Szabó and Khovanov have been known to give new lower bounds of $g_*^s(K)$.

It is well-known that lower bounds of the slice genus can be obtained through minimal genus problems in 4-manifolds with boundary; the slice genus of a knot K is bounded from below by the minimal genus for a specific homology class in the 4-manifold obtained by attaching a 2-handle to the 4-ball along K. It follows that Theorem 1.1 and Proposition 1.2 give lower bounds of the slice genus in terms of the ρ-invariants. In fact, it turns out that this method gives us lower bounds of the genus of a locally flat surface bounded by K in a homology 4-ball with boundary S^3. The following theorem illustrates that our lower bounds from the ρ-invariants actually reveal new information; one can detect arbitrarily large slice genus of knots that all the previously known lower bounds fail to detect.

Theorem 1.3. *For any positive integer g, there are infinitely many knots K with the following properties:*

(1) $g_*^t(K) = g_*^s(K) = g$.
(2) *K has a Seifert matrix of a slice knot.*
(3) *K has vanishing Casson-Gordon invariants.*
(4) *K has vanishing Ozsváth-Szabó τ-invariant and Rasmussen s-invariant.*

We remark that in the proof of Theorem 1.3 (1), $g_*^t(K)$ is detected by considering a minimal genus problem for which the results in [HS71, Rok71, LW97, LW00] give no interesting lower bound but the ρ-invariants give an optimal bound. We also remark that results of Cochran-Orr-Teichner [COT03] can be used to reveal partial information that $g_*^t(K) > 0$, i.e., K is not topologically slice.

As a consequence of Theorem 1.3 (4), it follows that the applications of the adjunction inequality to the smooth slice genus as in [Rud97, LM97, LM98, AM97] give us no information on K, since τ- and s-invariants are known to be sharper than the Thurston-Bennequin lower bound, due to Plamenevskaya [Pla04a, Pla04b] and Shumakovitch [Shu04]. The author knows no other method to apply gauge theory to estimate the slice genus of our K. Finally we remark that in the proof of Theorem 1.3 (4), we show a little more generalized statement (Lemma 5.4) that for any finitely collection $\{\Phi_\alpha\}$ of integer-valued homomorphisms of the smooth knot concordance group that give lower bounds of g_*^s, our K can be chosen in such a way that $\Phi_\alpha(K) = 0$ for each Φ_α, i.e., no such homomorphism extracts any information on the slice genus of K. For more detailed discussion on Theorem 1.3, see Section 5.

2. Betti numbers and L^2-signatures

In this section we prove Theorem 1.1 (1). The essential part of the proof is to estimate the L^2-Betti number of a 4-manifold in terms of the ordinary Betti number. From this the desired relationship between the ordinary Betti number and the L^2-signature follows, because L^2-dimension theory enables us to show that the L^2-signature is bounded by the (middle dimensional) L^2-Betti number; this is an L^2-analogue of a well-known fact that the ordinary signature is bounded by the Betti number. In this section all manifolds are topological manifolds.

Upper bounds of L^2-Betti numbers. We start by defining the algebraic L^2-Betti number. As a primary reference on the L^2-theory we need, we refer to Lück's book [Lüc02]. Let Γ be a discrete countable group. While L^2-invariants are usually defined via the *group von Neumann algebra* $\mathcal{N}\Gamma$, in this paper we will mainly use the *algebra* $\mathcal{U}\Gamma$ *of operators affiliated to* $\mathcal{N}\Gamma$, which is more useful for our purpose. Both coefficients are known to give the same L^2-Betti number and signature.

The L^2-dimension theory provides a dimension function

$$\dim^{(2)}_\Gamma\colon \{\text{finitely generated } \mathcal{U}\Gamma\text{-modules}\} \longrightarrow [0,\infty).$$

For a finite CW-complex X endowed with $\pi_1(X)\to\Gamma$, the twisted homology module

$$H_i(X;\mathcal{U}\Gamma) = H_i(C_*(X;\mathbb{Z}\Gamma)\otimes_{\mathbb{Z}\Gamma}\mathcal{U}\Gamma)$$

is defined by viewing $\mathcal{U}\Gamma$ as a $\mathbb{Q}\Gamma$-module via the natural inclusions $\mathbb{Q}\Gamma\to\mathcal{N}\Gamma\to\mathcal{U}\Gamma$, and is known to be finitely generated. The *L^2-Betti number* $\beta^{(2)}_i(X)$ is defined to be $\beta^{(2)}_i(X)=\dim^{(2)}_\Gamma H_i(X;\mathcal{U}\Gamma)$. For a CW-pair (X,A), $\beta^{(2)}_i(X,A)$ is similarly defined. (In this paper the choice of $\pi_1(X)\to\Gamma$ will always be clearly understood and so we do not include it in the notation.) It is known that the analytic and L^2-homological definitions are equivalent to the algebraic definition described here [Lüc02, Chapter 1, 6 and 8].

Following Cochran-Orr-Teichner [COT03], we will focus on the case of a *poly-torsion-free-abelian (PTFA) group*, which is defined to be a group admitting a finite length normal series $\{G_i\}$ with torsion-free abelian quotients G_i/G_{i+1}. In this paper Γ is always assumed to be PTFA. Also, we assume that $\pi_1(X)\to\Gamma$ is nontrivial, since a trivial homomorphism gives nothing beyond the (untwisted) rational coefficient.

Proposition 2.1. *Suppose W is a connected compact 4-manifold (possibly with nonempty boundary) endowed with a nontrivial homomorphism $\pi_1(W)\to\Gamma$. Then*

1. $\beta^{(2)}_1(W)\le\beta_1(W)-1$,
2. $\beta^{(2)}_2(W)\le\beta_2(W)$, *and*
3. $\beta^{(2)}_3(W)\le\begin{cases}\beta_3(W)-1 & \text{if } W \text{ is closed,}\\ \beta_3(W) & \text{otherwise.}\end{cases}$

Remark 2.2. (1) When ∂W is nonempty, the proposition also gives an upper bound of $\beta^{(2)}_i(W,\partial W)$ in terms of the ordinary Betti number, by duality.

(2) In the special case that $H_1(\partial W;\mathbb{Q})\cong H_1(W;\mathbb{Q})$ and ∂W is nonempty, a similar result was proved (at least implicitly) in [COT03]. Our proof of Proposition 2.1 proceeds similarly to [COT03], but we need some technical modification to get rid of the H_1-isomorphism condition.

Lemma 2.3 below provides facts on a PTFA group which are necessary to prove Proposition 2.1. For a proof of Lemma 2.3, see [COT03].

Lemma 2.3. (1) $\mathbb{Q}\Gamma$ *is an Ore domain so that there is a (skew-)field* $\mathcal{K}$ *of right quotients of* $\mathbb{Q}\Gamma$. *Every* $\mathcal{K}$*-module* M *is free and has a well-defined dimension* $\dim_{\mathcal{K}} M$.

(2) *Suppose that* C_* *is a finitely generated free chain complex over* $\mathbb{Q}\Gamma$. *If* $H_i(C_* \otimes_{\mathbb{Q}\Gamma} \mathbb{Q}) = 0$ *for* $i \le n$, *then* $H_i(C_* \otimes_{\mathbb{Q}\Gamma} \mathcal{K}) = 0$ *for* $i \le n$.

In particular, the existence of the skew-field $\mathcal{K}$ of quotients enables us to understand the L^2-dimension as the ordinary dimension over $\mathcal{K}$, as follows: it is known that if $\mathbb{Q}\Gamma$ is an Ore domain, then the natural map $\mathbb{Q}\Gamma \to \mathcal{U}\Gamma$ extends to an embedding $\mathcal{K} \to \mathcal{U}\Gamma$ [Lüc02]. For a space X equipped with $\pi_1(X) \to \Gamma$, let denote the Betti number with $\mathcal{K}$-coefficients by $\beta_i(X;\mathcal{K}) = \dim_{\mathcal{K}} H_i(X;\mathcal{K})$. By definition, $H_i(X;\mathcal{U}\Gamma)$ is the homology of the cellular chain complex

$$C_*(X;\mathbb{Q}\Gamma) \otimes_{\mathbb{Q}\Gamma} \mathcal{U}\Gamma = (C_*(X;\mathbb{Q}\Gamma) \otimes_{\mathbb{Q}\Gamma} \mathcal{K}) \otimes_{\mathcal{K}} \mathcal{U}\Gamma.$$

Since $H_*(X;\mathcal{K}) = H_*(C_*(X;\mathbb{Q}\Gamma) \otimes_{\mathbb{Q}\Gamma} \mathcal{K})$, we have the universal coefficient spectral sequence

$$E^2_{p,q} = \operatorname{Tor}^{\mathcal{K}}_p(H_q(X;\mathcal{K}), \mathcal{U}\Gamma) \Rightarrow H_{p+q}(X;\mathcal{U}\Gamma).$$

Since all higher Tor terms vanish over the $\mathcal{K}$-coefficient, it follows that

$$H_i(X;\mathcal{U}\Gamma) = H_i(X;\mathcal{K}) \otimes_{\mathcal{K}} \mathcal{U}\Gamma.$$

Therefore $H_i(X;\mathcal{U}\Gamma)$ is always a free $\mathcal{U}\Gamma$-module whose $\mathcal{U}\Gamma$-rank is equal to the $\mathcal{K}$-coefficient Betti number $\beta_i(X;\mathcal{K})$. Since $\dim^{(2)}_{\Gamma}(\mathcal{U}\Gamma)^n = n$ (e.g., see [Lüc02]), we obtain

Lemma 2.4. $\beta^{(2)}_i(X) = \beta_i(X;\mathcal{K})$, *and similarly for a pair* (X, A).

In order to prove Proposition 2.1, we first deal with the first Betti number.

Lemma 2.5. *Suppose* (X, A) *is a finite CW-pair with* X *connected, and* $\pi_1(X) \to \Gamma$ *is a homomorphism. Then*

(1) *If* A *is nonempty,* $\beta_1(X, A;\mathcal{K}) \le \beta_1(X, A)$.
(2) *If* A *is empty,* $\beta_1(X;\mathcal{K}) \le \beta_1(X) - 1$.

We remark that the absolute case (2) was shown in [COT03, Proposition 2.11].

Proof. Suppose that A is nonempty. Denote $\beta = \beta_1(X, A)$, and let (Y, B) be the disjoint union of β copies of $(I, \partial I)$ where $I = [0, 1]$. Choose a map $f: (Y, B) \to (X, A)$ which induced an isomorphism $H_1(Y, B;\mathbb{Q}) \to H_1(X, A;\mathbb{Q})$.

By replacing X with the mapping cylinder $M_f = (Y \times I) \cup X/(y, 0) \sim f(x)$ of f, and replacing A with $(B \times I) \cup A \subset M_f$, we may assume that f is an injection $(Y, B) \subset (X, A)$ and $Y \cap A = B$. From the homology long exact sequence with $\mathbb{Q}$-coefficients derived from

$$0 \longrightarrow C_*(Y, B) \longrightarrow C_*(X, A) \longrightarrow C_*(X, Y \cup A) \longrightarrow 0,$$

it follows that $H_i(X, Y \cup A;\mathbb{Q}) = 0$ for $i \le 1$. By Lemma 2.3 (2), $H_i(X, Y \cup A;\mathcal{K}) = 0$ for $i \le 1$. Thus, from the long exact sequence with $\mathcal{K}$-coefficients, it follows that f induces a surjection $H_1(Y, B;\mathcal{K}) \to H_1(X, A;\mathcal{K})$. This shows that $\beta_1(X, A;\mathcal{K}) \le \beta_1(Y, B;\mathcal{K})$. On the other hand, since $C_i(Y, B;\mathcal{K}) = 0$ for all i but $C_1(Y, B;\mathcal{K}) = \mathcal{K}^\beta$, $\beta_1(Y, B;\mathcal{K}) = \beta$. This completes the proof of (1).

Suppose A is empty. To apply the previous case, we choose a point $* \in X$ and consider the pair $(X, \{*\})$. In the exact sequence

$$0 \longrightarrow H_1(X;\mathcal{K}) \longrightarrow H_1(X, \{*\};\mathcal{K}) \longrightarrow H_0(\{*\};\mathcal{K}) \longrightarrow H_1(X;\mathcal{K}),$$

$H_0(\{*\};\mathcal{K}) = \mathcal{K}$ obviously and $H_0(X;\mathcal{K}) = \mathcal{K}/(\pi_1(X)\text{-action})$ is trivial since $\mathcal{K}$ is a division ring and $\pi_1(X) \to \Gamma$ is nontrivial. It follows that

$$\beta_1(X;\mathcal{K}) + 1 = \beta_1(X, \{*\};\mathcal{K}) \le \beta_1(X, \{*\}) = \beta_1(X). \qquad \square$$

Proof of Proposition 2.1. Suppose W is a compact connected 4-manifold equipped with $\pi_1(W) \to \Gamma$. Since W has the homotopy type of a finite CW-complex with cells of dimension ≤ 4, we may assume that the chain complex $C_*(W;-)$ is finitely generated and has dimension ≤ 4.

By Lemma 2.4, we can think of $\beta_i(W;\mathcal{K})$ instead $\beta_i^{(2)}(W)$. So (1) follows directly from Lemma 2.5.

To prove (3), observe that the duality implies $\beta_3(W;\mathcal{K}) = \beta_1(W,\partial W;\mathcal{K})$. If ∂W is empty, $\beta_1(W,\partial W;\mathcal{K}) \le \beta_1(W) - 1 = \beta_3(W) - 1$ by Lemma 2.5. If ∂W is nonempty, $\beta_1(W,\partial W;\mathcal{K}) \le \beta_1(W,\partial W) = \beta_3(W)$ again by Lemma 2.5.

To prove (2), we use the fact that the Euler characteristics for the $\mathbb{Q}$- and $\mathcal{K}$-coefficients are the same, that is,

$$\sum_{i=0}^{4}(-1)^i\beta_i(W;\mathcal{K}) = \sum_{i=0}^{4}(-1)^i\beta_i(W).$$

Since $\pi_1(W) \to \Gamma$ is nontrivial, $\beta_0(W;\mathcal{K}) = 0$. When W has nonempty boundary, $\beta_0(W,\partial W;\mathcal{K}) = 0$ since $\beta_0(W,\partial W;\mathcal{K}) \le \beta_0(W;\mathcal{K})$. From this it follows that $\beta_4(W;\mathcal{K}) = 0$. Plugging these values and the inequalities proved above into the Euler characteristic identity, we obtain $\beta_2(W;\mathcal{K}) \le \beta_2(W)$. □

Upper bounds of L^2-signatures. We define the *von Neumann L^2-signature* as follows: for a $4k$-manifold W endowed with $\pi_1(W) \to \Gamma$, the $\mathcal{U}\Gamma$-coefficient intersection form

$$\lambda\colon H_{2k}(W;\mathcal{U}\Gamma) \times H_{2k}(W;\mathcal{U}\Gamma) \longrightarrow \mathcal{U}\Gamma$$

is a hermitian form. In our case, $H_{2k}(W;\mathcal{U}\Gamma)$ is always a free $\mathcal{U}\Gamma$-module since Γ is assumed to be PTFA. By spectral theory, $H_{2k}(W;\mathcal{U}\Gamma)$ is decomposed as an orthogonal sum of canonically defined subspaces H_+, H_-, and H_0 such that λ is positive definite, negative definite, and trivial, on H_+, H_-, and H_0, respectively. The L^2-signature of W is defined to be

$$\operatorname{sign}^{(2)}(W) = \dim^{(2)}_\Gamma(H_+) - \dim^{(2)}_\Gamma(H_-) \in \mathbb{R}.$$

For more details and the relationship with other ways to define the L^2-signature, refer to [COT03] and [LS03].

Lemma 2.6. $|\operatorname{sign}^{(2)}(W)| \le \beta^{(2)}_{2k}(W)$.

Proof. Since $H_+, H_- \subset H_{2k}(W;\mathcal{U}\Gamma)$ and $H_+ \cap H_- = \{0\}$, L^2-dimension theory enables us to show

$$\dim^{(2)}_\Gamma(H_+) + \dim^{(2)}_\Gamma(H_-) \le \dim^{(2)}_\Gamma H_{2k}(W;\mathcal{U}\Gamma)$$

using an L^2-analogue of a standard argument of elementary linear algebra. (e.g., refer to Chapter 8 of [Lüc02], where it is shown that $\dim^{(2)}_\Gamma$ satisfies a set of axioms which includes all the properties we need.) From this the conclusion follows. □

Combining Lemma 2.6 with Lemma 2.1, we obtain:

Lemma 2.7. *If W is a compact connected 4-manifold endowed with a nontrivial homomorphism $\pi_1(W) \to \Gamma$, then*

$$|\operatorname{sign}^{(2)}(W)| \le \beta_2(W).$$

Now we are ready to show the first part of Theorem 1.1 stated in the introduction. We adopt the following topological definition of the ρ-invariant, as in Chang-Weinberger [CW03]. (See also Cochran-Orr-Teichner [COT03].) Let M be a 3-manifold endowed with $\pi_1(M) \to \Gamma$. It is known that there is a bigger group G containing Γ and a 4-manifold W such that ∂W consists of r components

$M_1, \ldots, M_r$ ($r > 0$), $M_i \cong M$, and $\pi_1(M_i) \xrightarrow{\phi} \Gamma \to G$ factors through $\pi_1(W)$ for each i. (For a proof, see the appendix of [CW03]; they consider the case that $\pi_1(M) = \Gamma$ but the same argument works in our case as well.) Then $\rho(M, \phi)$ is defined to be the following signature defect:

$$\rho(M, \phi) = \frac{1}{r}\big(\operatorname{sign}^{(2)}(W) - \operatorname{sign}(W)\big) \in \mathbb{R}$$

where $\operatorname{sign}^{(2)}(W)$ and $\operatorname{sign}(W)$ denote the L^2-signature associated to $\pi_1(W) \to G$ and the ordinary signature, respectively. The real number $\rho(M, \phi)$ is determined by M and ϕ, and independent of the choices we made. From the results in [LS03] it follows that $\rho(M, \phi)$ defined above coincides with the ρ-invariant of Cheeger-Gromov [CG85].

Proof of Theorem 1.1 (1). Suppose W is a compact connected 4-manifold with boundary M, and $\pi_1(W) \to \Gamma$ is given. Let denote the composition $\pi_1(M) \to \pi_1(W) \to \Gamma$ by ϕ. Our goal is to show that $|\rho(M, \phi)| \le 2\beta_2(W)$.

Since ϕ factors through $\pi_1(W)$, we can compute $\rho(M, \phi)$ using W; by the definition above,

$$\rho(M, \phi) = \operatorname{sign}^{(2)}(W) - \operatorname{sign}(W).$$

Obviously $|\operatorname{sign}(W)| \le \beta_2(W)$. By Lemma 2.7, $|\operatorname{sign}^{(2)}(W)| \le \beta_2(W)$. From this the desired conclusion follows. ☐

3. Extending coefficient systems to bounding 4-manifolds

Suppose W is a topological 4-manifold with boundary M and $\pi_1(W) \to \Gamma$ is given. (M is endowed with the induced map $\pi_1(M) \to \Gamma$.) In this section we deal with the problem of extending a bigger coefficient system on M to W to prove Theorem 1.1 (2). To state a more detailed form of Theorem 1.1 (2), we need the following facts from [COT03]: suppose $\mathcal{R}$ is a (possibly non-commutative) subring of $\mathcal{K}$ which is a PID containing $\mathbb{Z}\Gamma$. In this section we assume that $H_1(M; \mathcal{R})$ is $\mathcal{R}$-torsion.

(1) *Blanchfield form on $H_1(M; \mathcal{R})$.* The Bockstein map $B\colon H_2(M; \mathcal{K}/\mathcal{R}) \to H_1(M; \mathcal{R})$ and the Kronecker evaluation $\kappa\colon H^1(M; \mathcal{K}/\mathcal{R}) \to \operatorname{Hom}(H_1(M; \mathcal{R}), \mathcal{K}/\mathcal{R})$ are isomorphisms. The Blanchfield form, which is defined to be the isomorphism

$$B\ell\colon H_1(M; \mathcal{R}) \xrightarrow{B^{-1}} H_2(M; \mathcal{K}/\mathcal{R}) \xrightarrow{\text{duality}} H^1(M; \mathcal{K}/\mathcal{R}) \xrightarrow{\kappa} \operatorname{Hom}(H_1(M; \mathcal{R}), \mathcal{K}/\mathcal{R}),$$

is a symmetric linking form on $H_1(M; \mathcal{R})$ [COT03, p. 451].

(2) *Coefficient systems induced by characters.* A homomorphism $h\colon H_1(M; \mathcal{R}) \to \mathcal{K}/\mathcal{R}$ gives rise to a group homomorphism $\psi\colon \pi_1(M) \to \mathcal{K}/\mathcal{R} \rtimes \Gamma$ in a natural way. Indeed, ψ is a lift of $\pi_1(M) \to \Gamma$, i.e.,

$$\begin{tikzcd} & \mathcal{K}/\mathcal{R} \rtimes \Gamma \arrow[d, "\text{projection}"] \\ \pi_1(M) \arrow[ur, dotted, "\psi"] \arrow[r] & \Gamma \end{tikzcd}$$

commutes, and the restriction of ψ on $N = \operatorname{Ker}\{\pi_1(M) \to \Gamma\}$ agrees with

$$N \longrightarrow N/[N, N] = H_1(M; \mathbb{Z}\Gamma) \longrightarrow H_1(M; \mathcal{R}) \xrightarrow{h} \mathcal{K}/\mathcal{R} \subset \mathcal{K}/\mathcal{R} \rtimes \Gamma.$$

Furthermore, ψ factors through $\pi_1(W)$ if h factors through $H_1(W;\mathcal{R})$ [COT03, p. 455].

Note that $\mathcal{K}/\mathcal{R}$ is a torsion-free abelian group, and therefore $\mathcal{K}/\mathcal{R} \rtimes \Gamma$ is PTFA when Γ is PTFA. We also recall that, as in case of a commutative PID, any finitely generated $\mathcal{R}$-module M is isomorphic to $F \oplus tM$ where F is a free module of rank $\dim_{\mathcal{K}}(M \otimes_{\mathcal{R}} \mathcal{K})$ and tM is the $\mathcal{R}$-torsion submodule of M. (e.g., refer to [Coh71].) tM is isomorphic to a direct sum of cyclic modules of nonzero order.

Now we can state the result we will prove in this section. Denote by ∂ the boundary map $H_2(W, M;\mathcal{R}) \to H_1(M;\mathcal{R})$.

Theorem 3.1. *Suppose that $H_2(W, M;\mathcal{R}) = F \oplus tH_2(W, M;\mathcal{R})$ and $\partial(F)$ is a proper submodule of $H_1(M;\mathcal{R})$ for some free summand F. Then there is a nontrivial submodule P in $H_1(M;\mathcal{R})$ such that for any $x \in P$, the homomorphism*

$$B\ell(x)\colon H_1(M;\mathcal{R}) \longrightarrow \mathcal{K}/\mathcal{R}$$

factors through $H_1(W;\mathcal{R})$.

In particular, if $H_1(M;\mathcal{R})$ is never generated by $\beta_2(W)$ elements, then since

$$\beta_2(W, M;\mathcal{K}) = \beta_2(W;\mathcal{K}) = \beta_2^{(2)}(W) \le \beta_2(W)$$

by duality and Lemma 2.1, the hypothesis of Theorem 3.1 is satisfied. It it is the case, then for $x \in P$, $B\ell(x)$ gives rise to a homomorphism $\pi_1(M) \to \mathcal{K}/\mathcal{R} \rtimes \Gamma$ which factors through $\pi_1(W)$. This proves Theorem 1.1 (2).

The remaining part of this section is devoted to the proof of Theorem 3.1. As the first step, we will show that for the boundary of a relative 2-cycle of (W, M), the Blanchfield form of M can be computed via the intersection form of W. Indeed it is a consequence of the following algebraic observation:

Lemma 3.2. *Suppose $\mathcal{R}$ is a (possibly non-commutative) ring with (skew-)quotient field $\mathcal{K}$, and*

$$0 \longrightarrow C'_* \xrightarrow{i} C_* \xrightarrow{p} C''_* \longrightarrow 0$$

is an exact sequence of chain complexes over $\mathcal{R}$ such that $H_n(C' \otimes \mathcal{K}) = 0 = H_{n-1}(C' \otimes \mathcal{K})$. Then

$$\alpha\colon H_n(C''_*) \longrightarrow H_n(C''_* \otimes \mathcal{K}) \xrightarrow[\cong]{p_*^{-1}} H_n(C_* \otimes \mathcal{K}) \longrightarrow H_n(C_* \otimes \mathcal{K}/\mathcal{R})$$

coincides with

$$\beta\colon H_n(C''_*) \xrightarrow{\partial} H_{n-1}(C'_*) \xrightarrow[\cong]{B^{-1}} H_n(C'_* \otimes \mathcal{K}/\mathcal{R}) \xrightarrow{i_*} H_n(C \otimes \mathcal{K}/\mathcal{R}).$$

Proof. First note that the Bockstein B and the induced map p_* are isomorphisms since $H_n(C'_* \otimes \mathcal{K}) = 0 = H_{n-1}(C'_* \otimes \mathcal{K})$.

We will regard C'_* as a submodule of C_* and denote the homology class of a cycle x by $[x]$. Suppose z is a cycle in C''_n, and $x \in C_n$ is a pre-image of z, i.e., $p(x) = z$. $H_j(C'_*) \otimes \mathcal{K} = H_j(C'_* \otimes \mathcal{K}) = 0$ for $j = n$, $n-1$ since $\mathcal{K}$ is a flat $\mathcal{R}$-module, and therefore p induces an isomorphism $H_n(C_*) \otimes \mathcal{K} \cong H_n(C''_*) \otimes \mathcal{K}$. It follows that there is a cycle y in C_n such that $p_*[y] = [z] \cdot r$ in $H_n(C''_*)$ for some nonzero $r \in \mathcal{R}$, that is, there is $u \in C_{n+1}$ such that $\partial u = x \cdot r - y + w$ where $w \in C'_n \subset C_n$. Since $\partial y = 0$, $\partial w = \partial x \cdot r$. Therefore $w \otimes \frac{1}{r}$ is a cycle in $C'_n \otimes \mathcal{K}/\mathcal{R}$.

Since $p_*[y \otimes \frac{1}{r}] = [z \otimes 1]$, it can be seen that $\alpha[z] = [y \otimes \frac{1}{r}]$. On the other hand, by the definition of the Bockstein homomorphism, $B[w \otimes \frac{1}{r}] = [\partial x]$, and therefore, $\beta[z] = [w \otimes \frac{1}{r}]$.

In $C_n \otimes \mathcal{K}/\mathcal{R}$, we have

$$\partial\Big(u \otimes \frac{1}{r}\Big) = x - y \otimes \frac{1}{r} + w \otimes \frac{1}{r} = -y \otimes \frac{1}{r} + w \otimes \frac{1}{r}.$$

From this it follows that $[y \otimes \frac{1}{r}] = [w \otimes \frac{1}{r}]$ in $H_n(C_* \otimes \mathcal{K}/\mathcal{R})$. □

Recall that ∂ denotes the boundary map $H_2(W, M; \mathcal{R}) \to H_1(M; \mathcal{R})$.

Lemma 3.3. *Let Φ be the composition*

$$\begin{aligned}\Phi\colon H_2(W, M; \mathcal{R}) \longrightarrow H_2(W, M; \mathcal{K}) \cong H_2(W; \mathcal{K}) \longrightarrow H_2(W; \mathcal{K}/\mathcal{R})\\ \cong H^2(W, M; \mathcal{K}/\mathcal{R}) \xrightarrow{\kappa} \operatorname{Hom}(H_2(W, M; \mathcal{R}), \mathcal{K}/\mathcal{R}),\end{aligned}$$

where κ is the Kronecker evaluation map. Then $B\ell(\partial x)(\partial y) = \Phi(x)(y)$ for any $x, y \in H_2(W, M; \mathcal{R})$.

Proof. From Lemma 3.2 and the naturality of duality and the Kronecker evaluation, we obtain the following commutative diagram:

$$\begin{tikzcd}
 & H_2(W,M;\mathcal{R}) \arrow[d] \arrow[r, "\partial"] & H_1(M;\mathcal{R}) \arrow[dddd, bend left=60, "B\ell"] \\
H_2(W;\mathcal{K}) \arrow[r, "\cong"'] \arrow[d] & H_2(W,M;\mathcal{K}) & \\
H_2(W;\mathcal{K}/\mathcal{R}) \arrow[d, "\cong", "\text{duality}"'] & & H_1(M;\mathcal{K}/\mathcal{R}) \arrow[ll] \arrow[uu, "\cong"', "B"] \arrow[d, "\text{duality}", "\cong"'] \\
H^2(W,M;\mathcal{K}/\mathcal{R}) \arrow[d, "\kappa"'] & & H^1(M;\mathcal{K}/\mathcal{R}) \arrow[ll] \arrow[d, "\cong"', "\kappa"] \\
\operatorname{Hom}(H_2(W,M;\mathcal{R}),\mathcal{K}/\mathcal{R}) & & \operatorname{Hom}(H_1(M;\mathcal{R}),\mathcal{K}/\mathcal{R}) \arrow[ll, "\partial^{\#}"']
\end{tikzcd}$$

Here the map $\partial^{\#}$ is given by $\partial^{\#}(\psi)(y) = \psi(\partial y))$ for $\psi\colon H_1(M; \mathcal{R}) \to \mathcal{K}/\mathcal{R}$ and $y \in H_2(W, M; \mathcal{R})$. From this the conclusion follows. □

Proof of Theorem 3.1. For a submodule P in $H_1(M; \mathcal{R})$, we denote

$$P^\perp = \{y \in H_1(M; \mathcal{R}) \mid B\ell(x)(y) = 0 \text{ for all } x \in P\}.$$

Consider the exact sequence

$$\cdots \longrightarrow H_2(W; \mathcal{R}) \longrightarrow H_2(W, M; \mathcal{R}) \xrightarrow{\partial} H_1(M; \mathcal{R}) \longrightarrow H_1(W; \mathcal{R}) \longrightarrow \cdots.$$

We will show that there is a nontrivial submodule P in $H_1(M; \mathcal{R})$ such that the image $\partial(H_2(W, M; \mathcal{R}))$ is contained in $P^\perp$. Indeed from this claim it follows that, for any $x \in P$, $B\ell(x)\colon H_1(M; \mathcal{R}) \to \mathcal{K}/\mathcal{R}$ gives rise to a homomorphism $\operatorname{Coker} \partial \to \mathcal{K}/\mathcal{R}$, which automatically extends to $H_1(W; \mathcal{R})$ since $\mathcal{K}/\mathcal{R}$ is an injective $\mathcal{R}$-module. This completes the proof.

Recall that we wrote $H_2(W, M; \mathcal{R}) = F \oplus tH_2(W, M; \mathcal{R})$ where F is free and $tH_2(W, M; \mathcal{R})$ is the torsion submodule. To prove the claim, we consider the following two cases:

Case 1: Suppose $\partial(tH_2(W, M; \mathcal{R}))$ is nontrivial. Consider the composition Φ described in Lemma 3.3. For any $x \in tH_2(W, M; \mathcal{R})$ we have $\Phi(x) = 0$, since Φ factors through $H_2(W, M; \mathcal{K})$ which is torsion free. Therefore $B\ell(\partial x)(\partial y) = \Phi(x)(y) = 0$ for any $y \in H_2(W, M; \mathcal{R})$. This shows that $P = \partial(tH_2(W, M; \mathcal{R}))$ is a nontrivial submodule satisfying the desired property.

Case 2: Suppose $\partial(tH_2(W, M; \mathcal{R}))$ is trivial. Then the image of ∂ is equal to $\partial(F)$, which is a proper submodule of $H_1(M; \mathcal{R})$ by the hypothesis. Appealing to the lemma below, which should be regarded as folklore, it follows that $P = \partial(F)^\perp$ is nontrivial. It is obvious that this P has the desired property. □

Lemma 3.4. *Suppose A is a finitely generated torsion $\mathcal{R}$-module endowed with a symmetric linking form given by an isomorphism $\Psi\colon A \to \operatorname{Hom}(A, \mathcal{K}/\mathcal{R})$. Then for any proper submodule B in A, $B^\perp$ is nontrivial.*

Proof. From the exact sequence

$$0 \longrightarrow \operatorname{Hom}(A/B, \mathcal{K}/\mathcal{R}) \xrightarrow{p^\#} \operatorname{Hom}(A, \mathcal{K}/\mathcal{R}) \xrightarrow{i^\#} \operatorname{Hom}(B, \mathcal{K}/\mathcal{R})$$

it follows that $B^\perp = \Psi^{-1}(\operatorname{Ker} i^\#) = \Psi^{-1}(\operatorname{Im} p^\#)$. So it suffices to show that $\operatorname{Hom}(A/B, \mathcal{K}/\mathcal{R})$ is nontrivial. Note that every cyclic module $\mathcal{R}/p\mathcal{R}$ with $p \ne 0$ is (isomorphic to) a submodule of $\mathcal{K}/\mathcal{R}$. Since A/B is a nontrivial torsion module, it has a summand of the form $\mathcal{R}/p\mathcal{R}$ with $p \ne 0$, by the structure theorem of finitely generated $\mathcal{R}$-modules. It follows that $\operatorname{Hom}(A/B, \mathcal{K}/\mathcal{R})$ is nontrivial. □

4. Construction of a bordism from a locally flat surface

In this section we will prove Proposition 1.2. Suppose W is a topological 4-manifold with boundary M such that $H_1(W) = 0$, and σ is a 2-dimensional homology class contained in $\operatorname{Im}\{H_2(M) \to H_2(W)\}$. First we describe a homomorphism $\phi_\sigma\colon \pi_1(M) \to \mathbb{Z}$ which is determined by σ. Consider the exact sequence

$$H_2(W) \longrightarrow H_2(W, M) \xrightarrow{\partial} H_1(M) \longrightarrow H_1(W) = 0.$$

The intersection with σ gives a homomorphism $\sigma\cdot\colon H_2(W, M) \to \mathbb{Z}$, which induces a homomorphism $h_\sigma\colon H_1(M) \to \mathbb{Z}$ since $\sigma\cdot$ vanishes on the image of $H_2(W)$. Define ϕ_σ to be the composition

$$\phi_\sigma\colon \pi_1(M) \longrightarrow H_1(M) \xrightarrow{h_\sigma} \mathbb{Z}.$$

Recall that Proposition 1.2 claims that if there is a locally flat surface Σ of genus g in W which represents the class $\sigma \in H_2(W)$ and the map ϕ_σ is nontrivial, then there is a topological 4-manifold V bounded by M such that $\beta_2(V) = \beta_2(W) + 2g - 1$ and ϕ_σ factors through $H_1(V)$. Roughly speaking, we will construct V by performing "surgery along Σ" on W.

Proof of Proposition 1.2. By Alexander duality, $H_2(W, W - \Sigma)$ can be identified with $H^2(\Sigma) = \mathbb{Z}$. From the exact sequence

$$H_2(W) \xrightarrow{\sigma\cdot} H_2(W, W - \Sigma) \longrightarrow H_1(W - \Sigma) \longrightarrow H_1(W) = 0$$

it follows that $H_1(W - \Sigma) \cong H_2(W, W - \Sigma) = \mathbb{Z}$ since the leftmost map $\sigma\cdot$ is given by the intersection of a 2-cycle with σ, which is always zero.

Note that Σ has trivial normal bundle in W since Σ is connected and the self-intersection $\sigma \cdot \sigma$ vanishes. There is a bijection between the set of (fiber homotopy classes of) framings on Σ and $[\Sigma, S^1] = H^1(\Sigma, \mathbb{Z})$ which can be identified with $\mathbb{Z}^{2g}$ by choosing a basis $\{x_i\}$ of $H_1(\Sigma)$. Pushoff along a framing induces a homomorphism $H_1(\Sigma) \to H_1(W - \Sigma)$ in such a way that if the framing corresponding to $0 \in \mathbb{Z}^{2g}$ induces $h\colon H_1(\Sigma) \to H_1(W - \Sigma)$, then the framing corresponding to $(a_i) \in \mathbb{Z}^{2g}$ gives rise to a homomorphism sending x_i to $h(x_i) + a_i[\mu]$ where μ is a meridional curve of Σ. Since $H_1(W - M) \cong \mathbb{Z}$ is generated by $[\mu]$, it follows that there is a framing inducing a trivial homomorphism $H_1(\Sigma) \to H_1(W - \Sigma)$. We identify a tubular neighborhood of Σ in W with $\Sigma \times D^2$ under this framing, and denote $N = W - \operatorname{int}(\Sigma \times D^2)$.

Choose a 3-manifold R with boundary Σ such that $H_1(\Sigma) \to H_1(R)$ is surjective (e.g., a handlebody with the same genus as Σ may be used as R). Let

$$V = \big(N \cup (R \times S^1)\big)/\sim$$

where $\Sigma \times S^1 \subset \partial N$ and $\partial R \times S^1$ are identified. From the Mayer-Vietoris sequence

$$\cdots \longrightarrow H_1(\Sigma \times S^1) \longrightarrow H_1(N) \oplus H_1(R \times S^1) \longrightarrow H_1(V) \longrightarrow 0$$

for $V = N \cup (R \times S^1)$, it follows that $H_1(V) \cong H_1(N) = \mathbb{Z}$ since $H_1(\Sigma) \to H_1(R)$ is surjective and $i_*\colon H_1(\Sigma) \to H_1(N)$ is trivial by our choice of the framing on Σ. From the definition it is easily seen that h_σ is equal to the map $H_1(M) \to H_1(V) = \mathbb{Z}$ induced by the inclusion. Therefore ϕ_σ factors through $\pi_1(V)$ as desired.

The Betti number assertion follows from a straightforward computation. For the convenience of the reader, we give details below. From the above Mayer-Vietoris sequence it follows that

$$\chi(\Sigma \times S^1) + \chi(V) = \chi(N) + \chi(R \times S^1)$$

where χ denotes the Euler characteristic. $\chi(N) + \chi(\Sigma) = \chi(W)$ by the long exact sequence for the pair (W, N) and Alexander duality. Since $\chi(X \times S^1) = 0$ for any X, it follows that

$$\chi(V) = \chi(W) - \chi(\Sigma) = \chi(W) + 2g - 2.$$

From the hypothesis that $H_1(W) = 0$, it follows that $\beta_1(W) = 0$ and $\beta_3(W) = \beta_1(W, M) = \beta_0(M) - 1$. $\beta_1(V) = 1$ as shown above. Since ϕ_σ is nontrivial, so is $H_1(M) \to H_1(V) = \mathbb{Z}$ and thus has torsion cokernel. It follows that $\beta_3(V) = \beta_1(V, M) = \beta_0(M) - 1$. Combining these observations on the Betti numbers with the Euler characteristic identity, the desired inequality follows. □

5. Slice genus

In this section we apply the results proved in the previous sections to investigate the slice genus of a knot K in S^3. Indeed our results give lower bounds of the genus of a spanning surface in a homology 4-ball; for a knot K in a homology 3-sphere Y which bounds some (topological) homology 4-ball, let $g_*^h(K)$ be the minimal genus of a locally flat surface F in a homology 4-ball X such that $\partial(X, F) = (Y, K)$. Obviously $g_*^h(K) \le g_*^t(K) \le g_*^s(K)$ for a knot K in S^3.

For (X, F) as above, consider the 4-manifold W obtained by attaching a 2-handle to X along the preferred framing of K. The boundary of W is the result of surgery on Y along the preferred framing of K, which we will call the *zero-surgery manifold of* K and denote by M_K. Note that $H_1(M_K) = \mathbb{Z}$ is generated by a meridian of K. Let σ be a generator of $H_2(W) = \mathbb{Z}$. It can be easily seen that the abelianization map $\phi\colon \pi_1(M_K) \to H_1(M_K) = \mathbb{Z}$ is exactly the homomorphism ϕ_σ defined in Section 4. Also, note that σ is represented by a surface in M_K, namely a capped-off Seifert surface of K.

Attaching to F the core of the 2-handle of W, we obtain a surface Σ with the same genus as F which represents the homology class $\sigma \in H_2(W)$. Therefore, by Proposition 1.2, one obtains a null-bordism of M_K over $\mathbb{Z}$ with bounded β_2; we state it as a proposition.

Proposition 5.1. *There is a topological 4-manifold V with boundary M_K such that $\phi\colon \pi_1(M_K) \to \mathbb{Z}$ factors through $\pi_1(V)$ and $\beta_2(V) \le 2g_*^h(K)$. In particular, if K is a knot in S^3, then $\beta_2(V) \le 2g_*^t(K)$.*

This enables us to use Theorem 1.1, possibly repeatedly, to obtain lower bounds of $g_*^h(K)$. We remark that while a lower bound is obtained from $\rho(M_K, \phi)$ by applying Theorem 1.1 (1) directly, it gives us no interesting result since it is known that $\rho(M_K, \phi)$ is determined by the signature function of K [COT03]. However, it turns out that the ρ-invariants associated to bigger coefficient systems obtained by Theorem 1.1 (2) actually reveal new information on the slice genus which cannot be obtained via previously known invariants, as mentioned in Theorem 1.3. The

remaining part of this section is devoted to the construction of examples illustrating this.

Construction of examples. Our examples will be constructed using a well known method that produces a new knot from a given knot by "tying" another knot along a circle in the complement. For a knot J, we denote its exterior by $E_J = S^3-$ (open tubular neighborhood of J). Suppose K_0 is a knot and η is a circle in $S^3 - K_0$ which is unknotted in S^3. Choose a (closed) tubular neighborhood U of η in $S^3 - K_0$. Removing the interior of U from $S^3 - K_0$ and attaching the exterior E_J of a knot J along the boundary of U in such a way that a meridional curve of η is identified with a curve null-homologous in E_J, one obtains the complement of a new knot in S^3, which we will denote by $K_0(\eta, J)$. In some literature this construction is called the "satellite construction" or "genetic infection".

We start by choosing a knot K_s in S^3 whose Alexander polynomial $\Delta_{K_s}(t)$ is a cyclotomic polynomial $\Phi_n(t)$ with n divisible by at least three distinct primes. Indeed, by a well-known characterization due to Levine, there is such a knot if and only if $\Phi_n(t^{-1}) = \pm t^s \Phi_n(t)$ for some s and $\Phi_n(1) = \pm 1$. Since the complex conjugate of a root of unity is its reciprocal, $\Phi_n(t)$ satisfies the former condition. For the latter condition, one may appeal to the following lemma:

Lemma 5.2. *For $n \geq 2$, $\Phi_n(1) = 1$ if and only if n is not a prime power.*

Proof. If $n = p^a$ is a prime power, then it is easily seen that $\Phi_n(t)$ is given by

$$\Phi_n(t) = t^{p^{a-1}(p-1)} + \cdots t^{p^{a-1}} + 1$$

and therefore $\Phi_n(1) = p$.

Conversely, suppose $n = p_1^{a_1} \cdots p_r^{a_r}$ with p_i prime and $r > 1$. We recall that

$$t^n - 1 = \prod_{d|n} \Phi_d(t).$$

By eliminating the factor of $t - 1$ and rearranging terms, we obtain

$$t^{n-1} + \cdots + t + 1 = \left(\prod_{i=1}^{r} \prod_{j=1}^{a_i} \Phi_{p_i^j}(t) \right) \cdot \Phi_n(t) \cdot h(t)$$

Plugging $t = 1$, it follows that $\Phi_n(1)h(1) = 1$ and so $\Phi_n(1) = 1$. $\square$

Denote the (rational) Alexander module $H_1(M_J; \mathbb{Q}[t, t^{-1}])$ of a knot J by A_J, and the mirror image of J by $-J$. (Here we adopt the standard convention of the orientation of $-J$ so that $J\#(-J)$ is always a ribbon knot.)

Returning to our construction, for an unknotted circle in η disjoint to K_s and two knots J and J' which will be chosen later, consider the connected sum

$$K = \overset{g}{\#} \left(K_s(\eta, J) \# - (K_s(\eta, J')) \right)$$

of g identical knots.

We choose η in such a way that the following properties are satisfied:

- (P1) The linking number of η and K_s vanishes, so that η represents a homology class $[\eta] \in A_{K_s}$. Furthermore, $[\eta]$ is a generator of A_{K_s}.
- (P2) For any J and J', K satisfies $g_*^s(K) \leq g$.
- (P3) For any J and J', K is algebraically slice, i.e., K has a Seifert matrix of a slice knot.
- (P4) For any J and J', K has vanishing Casson-Gordon invariants.

For this purpose, we first choose a Seifert surface F of K_s. F consists of one 0-handle and $2r$ 1-handles, where r is the genus of F. Choose unknotted circles $\gamma_1, \ldots, \gamma_{2r}$ in $S^3 - F$ which are Alexander dual to the 1-handles of F, as illustrated in Figure 1.

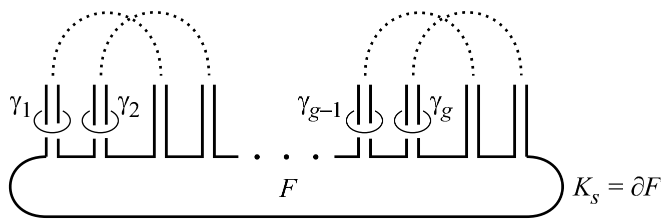

FIGURE 1.

Since each γ_i is disjoint to F, it represents a homology class $[\gamma_i] \in A_{K_s}$. Also, it can be seen that the $[\gamma_i]$ generate A_{K_s}, by a standard Mayer-Vietoris argument. Therefore one of the $[\gamma_i]$, say $[\gamma_1]$, is nontrivial in A_{K_s}. Let η be γ_1.

Lemma 5.3. *η satisfies the properties (P1)–(P4) required above.*

Proof. Obviously η has linking number zero with K. Since $\Delta_{K_s}(t)$ is irreducible, $A_{K_s} = \mathbb{Q}[t, t^{-1}]/\langle \Delta_{K_s}(t) \rangle$, and $[\eta] \neq 0$ is automatically a generator of A_{K_s}. This shows (P1).

Let $L = K_s(\eta, J)\# - (K_s(\eta, J'))$. We claim that $g_*^s(L) \leq 1$, from which (2) easily follows. To prove the claim, observe that L is obtained from the ribbon knot $K_s\#(-K_s)$, by "tying" J and J'. Note that the boundary connected sum of F and $-F$ is a Seifert surface for $K_s\#(-K_s)$. Tying J and J', the Seifert surface of $K_s\#(-K_s)$ becomes a Seifert surface E of genus $2r$ for L. E consists of a single 0-handle and $4r$ 1-handles $H_1, \ldots, H_{4r}$, where H_i is the image of the H_{4r-i+1} under an obvious reflection, for $2 \leq i \leq 2r$. Joining the endpoints of the core of H_i to their image under the reflection using disjoint arcs on the 0-handle of E for $2 \leq i \leq 2r$, we obtain $(2r-1)$ disjoint circles $\alpha_2, \ldots, \alpha_{2r}$ on E. See Figure 2.

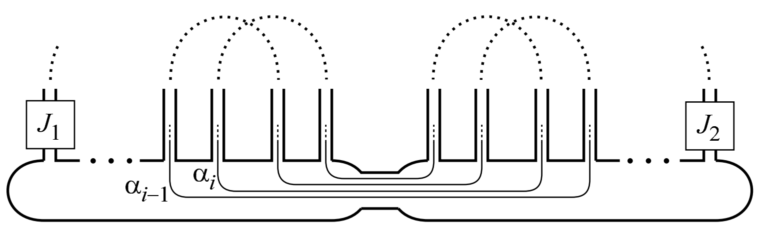

FIGURE 2.

The union of the α_i is a smoothly slice link, being the connected sum of a link and its mirror image. Thus there are disjoint 2-disks $D_1, \ldots, D_{2r-1}$ smoothly embedded in D^4 such that $\partial D_i = \alpha_i$. Since the Seifert form defined on E vanishes at (α_i, α_j), one can do ambient surgery on E along the α_i, using the disks D_i in D^4, as in [Lev69]. This produces a genus one surface in D^4 with boundary L. Therefore $g_*^s(L) \leq 1$. This completes the proof of (P2).

Since L shares a Seifert matrix with $K_s\#(-K_s)$ which is a ribbon knot, L is algebraically slice. From this (P3) follows.

It is easily seen that $\Delta_K(t) = \Phi_n(t)^{2g}$. Since n has been chosen to be divisible by three distinct primes, (P4) holds due to a result of Livingston [Liv02]. $\square$

Let $\mathcal{C}$ be the smooth knot concordance group.

Lemma 5.4. *Suppose $\{\Phi_\alpha\colon \mathcal{C} \to \mathbb{Z}\}$ is a finite collection of group homomorphisms satisfying $|\Phi_\alpha(-)| \le f_\alpha(g^s_*(-))$ for some real-valued function f_α. Then, there are knots J and J' such that our K satisfies the followings:*

(1) $g^h_*(K) = g^t_*(K) = g^s_*(K) = g$.
(2) $\Phi_\alpha(K) = 0$ *for each* Φ_α.

Note that the Ozsváth-Szabó τ-invariant [OS03] and Rasmussen s-invariant [Ras04] can be viewed as homomorphisms of $\mathcal{C}$ giving lower bounds of g^s_*. Therefore, from Lemma 5.4 (2), it follows that J and J' can be chosen in such a way that K has vanishing τ- and s-invariants.

Proof of Lemma 5.4. Let K' be the connected sum of g copies of $K_s\#(-K_s)$. By Cheeger-Gromov [CG85], there is a universal bound C of the ρ-invariants of the zero-surgery manifold $M_{K'}$ of K', i.e., $|\rho(M_{K'}, \phi')| \le C$ for any homomorphism ϕ' of $\pi_1(M_{K'})$.

Following [COT04], for a knot J, let

$$\rho(J) = \int_{S^1} \sigma_J(\omega)\, d\omega$$

be the integral of the knot signature function

$$\sigma_J(\omega) = \operatorname{sign}((1-\omega)S + (1-\bar\omega)S^T)$$

over the unit circle S^1 normalized to unit length, where S is a Seifert matrix of J.

We claim that there are two knots J and J' such that

(i) $|\rho(J)| \ge C + 4g$,
(ii) $|\rho(J')| \ge C + 4g + g\cdot|\rho(J)|$, and
(iii) $\Phi_\alpha(K_s(\eta, J)) = \Phi_\alpha(K_s(\eta, J'))$ for each Φ_α.

To prove the claim, we consider a sequence $\{J_i\}$ of knots constructed inductively as follows. Let J_0 be a knot with $|\rho(J_0)| \ge C + 4g$. Assuming J_i has been chosen, let J_{i+1} be a knot satisfying

$$|\rho(J_{i+1})| \ge C + 4g + g\cdot|\rho(J_i)|.$$

For example, one can choose as J_i the connected sum of sufficiently many copies of any knot with nonvanishing ρ, e.g., the trefoil knot, since ρ is additive under connected sum.

Since

$$g^s_*(K_s(\eta, J_i)) \le g(K_s(\eta, J_i)) \le g(K_s)$$

where $g(-)$ denotes the 3-genus (Seifert genus), there is an upper bound, say M_α, of $f_\alpha\big(g^s_*(K_s(\eta, J_i))\big)$, i.e., $f_\alpha\big(g^s_*(K_s(\eta, J_i))\big) \le M_\alpha$ for any J_i. Since

$$|\Phi_\alpha(K_s(\eta, J_i))| \le f_\alpha\big(g^s_*(K_s(\eta, J_i))\big)$$

by our hypothesis, it follows that $|\Phi_\alpha(K_s(\eta, J_i))|$ is bounded by M_α. Therefore the function $\mathbb{Z} \to \mathbb{Z}^{|\{\Phi_\alpha\}|}$ given by

$$i \longrightarrow (\Phi_\alpha(K_s(\eta, J_i)))_\alpha$$

has finite image. It follows that for some $i < j$, $\Phi_\alpha(K_s(\eta, J_i)) = \Phi_\alpha(K_s(\eta, J_j))$ for each Φ_α. Choosing $J = J_i$ and $J' = J_j$, the claim follows. (Indeed our argument shows that there are infinitely many pairs (J, J') satisfying the desired properties.)

Recall that our K is given by

$$K = \overset{g}{\#}\big(K_s(\eta, J)\# - (K_s(\eta, J'))\big).$$

By (iii) above, Φ_α vanishes at $K_s(\eta, J)\# - (K_s(\eta, J'))$. It follows that $\Phi_\alpha(K) = 0$ for each Φ_α. This proves the second conclusion of the lemma.

To prove the first conclusion, it suffices to show that $g_*^h(K) \geq g$ by the property (P2) above. Suppose $g_*^h(K) < g$. By Proposition 5.1, there is a 4-manifold V bounded by M_K such that $\beta_2(V) < 2g$ and $\phi\colon \pi_1(M_K) \to \mathbb{Z}$ factors through $\pi_1(V)$.

Letting $\Gamma = \mathbb{Z}$, $\mathcal{R} = \mathbb{Q}[t, t^{-1}]$, and $\mathcal{K} = \mathbb{Q}(t)$, we will apply Theorem 1.1 (2) to obtain a new coefficient system ϕ_1 which is a lift of ϕ. The conditions required in Theorem 1.1 (2) are verified as follows. It is well-known that $A_K = H_1(M_K; \mathcal{R})$ is always $\mathcal{R}$-torsion. We claim that A_K is not generated by $\beta_2(V)$ elements. Since the Alexander module is additive under connected sum and the knots $K_s(\eta, J)$ and $K_s(\eta, J')$ share the Alexander module with K_s, we have $A_K = \bigoplus^{2g} A_{K_s}$. Since A_{K_s} is nontrivial and $\beta_2(V) < 2g$, A_K is never generated by $\beta_2(V)$ elements as claimed, by appealing to the structure theorem of finitely generated modules over $\mathbb{Q}[t, t^{-1}]$.

Therefore, by applying Theorem 1.1 (2) and then (1), it follows that there is a nontrivial homomorphism $h\colon A_K \to \mathcal{K}/\mathcal{R}$ that gives rise to a homomorphism

$$\phi_1 \colon \pi_1(M_K) \longrightarrow (\mathcal{K}/\mathcal{R}) \rtimes \Gamma$$

such that

$$(*) \qquad |\rho(M_K, \phi_1)| \leq 2\beta_2(V) < 4g.$$

Note that K can be viewed as a knot obtained from K' by tying J and $-J'$ g times. So, from [COT04, Proposition 3.2] it follows that for some $\phi' \colon \pi_1(M_{K'}) \to (\mathcal{K}/\mathcal{R}) \rtimes \Gamma$,

$$\rho(M_K, \phi_1) = \rho(M_{K'}, \phi') + \sum_{i=1}^{g} n_i \rho(J) - \sum_{i=1}^{g} m_i \rho(J').$$

Here $n_i = 0$ if the $(2i-1)$-st factor of $A_K = \bigoplus^{2g} A_{K_s}$ is contained in the kernel of h, and $n_i = 1$ otherwise. The m_i are determined similarly by the behaviour of the $(2i)$-th factor of A_K.

Since h is a nontrivial homomorphism of A_K, at least one n_i or m_i is nonzero. If $m_i = 0$ for all i, then since $n_i \neq 0$ for some i, we have

$$|\rho(M_K, \phi_1)| \geq |\rho(J)| - |\rho(M_{K'}, \phi')| \geq (4g + C) - C = 4g$$

by (i) above. It contradicts $(*)$. Therefore $m_i \neq 0$ for some i. In this case, by (ii) above, we have

$$|\rho(M_K, \phi_1)| \geq |\rho(J')| - g \cdot |\rho(J)| - |\rho(M_{K'}, \phi')| \geq (4g + C) - C = 4g.$$

It again contradicts $(*)$. This shows that $g_*^h(K) \geq g$. □

Remark 5.5. It can be easily seen that our construction produces infinitely many knot types of K. In fact, one can use infinitely many knot types as our K_s, J, and J'.

Remark 5.6. Using results in [COT03], it can be shown that the nonvanishing of the ρ-invariants we considered in the proof of Lemma 5.4 implies that our K is not topologically slice. Our result generalizes this. In fact, our construction can be used to construct K which is (1)-solvable but not (1.5)-solvable, in the sense of [COT03]. It would be an interesting question whether there are (h)-solvable knots with topological slice genus g for any $h \in \frac{1}{2}\mathbb{Z}$ and any $g > 1$.

We finish this section with an observation on the failure of an attempt to extract information on the minimal genus for our example using previously known results. In [KM61, HS71, Rok71, LW97, LW00] lower bounds of the topological minimal genus are obtained for a homology class $\sigma \in H_2(X)$ in a topological 4-manifold X which is closed or has boundary consisting of homology sphere components. When X is simply connected, [KM61] provides an obstruction to being represented

by a locally flat sphere, i.e., minimal genus ≥ 1, based on the Rokhlin theorem. When $H_1(X) = 0$, [HS71, Rok71, LW97, LW00] provides higher lower bounds of the following form:

$$2 \cdot (\text{minimal genus}) \geq -\beta_2(X) + \max_{0\leq j<d} \left| \operatorname{sign}(X) - \frac{2j(d-j)}{d^2}(\sigma\cdot\sigma) \right|$$

where d is a positive integer such that σ is contained in the subgroup $d \cdot H_2(X)$. (A more refined result of Lee-Wilczyński [LW00, Theorem 2.1] may potentially give further lower bounds, however, computation seems infeasible when $H_1(X) \neq 0$.)

For an arbitrary 4-manifold W with boundary and $\sigma \in H_2(W)$, if W embeds into a 4-manifold X such that the above inequality gives a lower bound for the image of σ in X, then the result is also a lower bound for σ in W. However, when the self-intersection of σ is trivial, the above inequality gives no information since $\beta_2(X) \geq |\operatorname{sign}(X)|$ for any X. In particular, in the 4-manifold W obtained by attaching a 2-handle to the 4-ball along the zero-framing of the knot K constructed in this section, the generator $\sigma \in H_2(W)$ has vanishing self-intersection so that the minimal genus is not detected in this way. Our results show that the minimal genus for σ in W is exactly g.

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